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# ANALYSIS OF LOSSES IN ASW DEFENCE OF SHIPPING CAMPAIGNS

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La Spezia, Italy

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# SACLANT ASW RESEARCH CENTRE MEMORANDUM

## ANALYSIS OF LOSSES IN ASW DEFENCE OF SHIPPING CAMPAIGNS

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This Memorandum has been prepared within the Force Effectiveness Studies Group and does not necessarily represent the considered opinion of the SACLANT ASW Research Centre, of SACLANT, or of NATO.

R Nagelhout

R. Nagelhout
Group Leader

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# ANALYSIS OF LOSSES IN ASW DEFENCE-OF-SHIPPING CAMPAIGNS

by

Robert R.V. Wiederkehr

## ABSTRACT

Hierarchies of models have been developed that can be used to describe and analyse the losses resulting from large-scale ASW defence-of-shipping campaigns. The models take account of many different categories of merchant ships and submarines operating in different geographical areas. The effects of direct and indirect protection, submarine deployments, cycle time, endurance, weapon load, and duration of attack are included. The hierarchy of models provides a convenient means for determining where and why losses differ; it is also useful for conducting sensitivity analyses and appreciating how measures of effectiveness at the tactical level are related to those at the campaign level.

## INTRODUCTION

In assessing the effectiveness of alternative methods of sailing and protecting merchant shipping in a defence-of-shipping campaign, it is desirable to have models for estimating losses to merchant ships, ASW units, enemy submarines, etc. Models of this kind have been developed at SACLANTCEN and are reported in Refs. 1, 2 and 3.

In attempting to apply these models to realistic scenarios involving several different classes of submarines deployed in different operating areas, and different classes of merchant ships sailing on several different trade routes, it became clear that the computational effort required to employ these models would be excessive. Efforts were then directed towards simplifying these models without destroying the essential relationship between important inputs and outputs.

The first simplification to be introduced was to consider only the expected losses of each type of merchant ships, submarines, etc. This simplification is justified in Ref. 2, where it is shown that, for the stochastic process considered, the expected value of the submarine losses is equal to the variance of the submarine losses and that the expected value of the merchant ship losses is somewhat smaller than the variance of the merchant ship losses. Knowledge of the expected losses, therefore, gives one an indication of the variance of the losses as well.

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## ACKNOWLEDGEMENT

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The second simplification to be introduced concerns the time dependency of the engagement outcomes and engagement rates per submarine over the course of a defence-of-shipping campaign. A detailed accounting of this time dependency requires that sets of differential equations be solved numerically using a procedure such as that given in Ref. 4. In applying this procedure to a defence-of-shipping situation considered to be fairly realistic, except that a single shipping rate was considered [see Ref. 5], it was discovered that parameters describing engagement outcomes changed very little over the course of the battle. It appears quite reasonable, therefore, to consider these parameters to be constants over the course of the battle. Furthermore, for the situation considered in this report — where the number of merchant ships available is sufficiently large to permit a specified delivery rate to be achieved — it is reasonable to assume that the engagement rate per submarine is also constant over the course of the battle.

On the basis of the later simplifications, it is possible to solve sets of simultaneous differential equations and arrive at algebraic expressions for the expected losses of several types of merchant ships, escorts and submarines. These expressions can be factored into terms that have clear physical interpretations, such as the expected number of patrols per submarine and the number of engagements per submarine patrol.

The algebraic models have been used not only for estimating the losses of various types of units involved in defence-of-shipping campaigns, but also for explaining and comparing results corresponding to different conditions, such as variations in convoy size, shipping patterns, tactics, etc*. An additional use for these algebraic models is to perform a hierarchy of sensitivity analyses, a notion that is discussed in more detail in Ch. 4.

## 1. THE SITUATION

The situation envisaged in this study is similar to that described in Refs. 2, 3 and 5 and is described briefly as follows. Merchant-ship convoys and independents transport cargo across the ocean where they are subjected to attack by enemy submarines. These submarines, which cycle between their home port (or another replenishment area) and the anti-shipping operating area, are subjected to attrition either by indirect defences — such as AS barriers — or by direct defences — such as surface and air screens protecting the convoy.

In contrast to Refs. 2 and 3 there may be several classes of submarines deployed in several different anti-shipping operating areas, and several different types of ships sailing on many different trade routes. In contrast to Ref. 3, where it was assumed that there were insufficient ships to sustain the desired cargo delivery rate, it is assumed here that, even when ship losses are taken into account, there are enough ships available to sustain this desired rate.

In consonance with Refs. 2, 3 and 5, the time on patrol of submarines is limited either by torpedo load (which translates into a maximum number of attacks) or by the endurance of the submarine.

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* *These classified results are reported elsewhere.*

## 2. ATTRITION MODEL

### 2.1 Categories of Merchant Ships and Submarines

The broad categories of units that may suffer losses in the above situation are: merchant ships, escorts and submarines. However, a finer subdivision of categories is desirable for examining the performance of convoys and independents travelling on different routes at different speeds with different levels of direct protection, and of submarines of different classes operating in different patrol areas.

A general way of accounting for the various categories that arise in any specific problem is to label each category of merchant ship with an index i and each category of submarine with an index j. A simple example of this indexing procedure is given below:

| Index i | Merchant Ship Categories |
|---|---|
| 1 | Fast convoys, Northern route |
| 2 | Slow convoys, Northern route |
| 3 | Independents, Northern route |
| 4 | Fast convoys, Southern route |
| 5 | Slow convoys, Southern route |
| 6 | Independents, Southern route |

| Index j | Submarine Categories |
|---|---|
| 1 | Nuclear, Western operating area |
| 2 | Nuclear, Eastern operating area |
| 3 | Conventional, Western operating area |
| 4 | Conventional, Eastern operating area |

The total number of merchant ship categories will be denoted by $I$ and the total number of submarine categories will be denoted by $J$. In the above example $I = 6$ and $J = 4$. In a realistic scenario values of $I$ and $J$ may be 30 or more.

### 2.2 Loss-Rate Equations and their Solution

The stochastic analysis of Ref. 2 produced differential equations for the expected value and variance of the merchant ship and submarine losses. Using the notion developed in Appendix A, it can be shown (see App. B) that the differential equation for the expected merchant ship losses of Ref. 2 has the following simple interpretation.

The expected merchant-ship loss rate equals the product of three quantities:

1. The engagement rate for disengaged submarine on patrol.
2. The number of disengaged submarines on patrol.
3. The expected number of merchant ships lost per engagement.

By analogy one can obtain similar equations for expected escort loss rates and expected submarine loss rates. This has been done in Appendix B.

Under the assumption of constant engagement rates per submarine and constant engagement outcomes (such as the expected number of merchant ships lost per engagement), the resulting set of differential equations can be solved and factored into terms that have the following clear physical interpretations (see App. B):

The expected number of merchant ships of category i lost in a battle of duration t as a result of engagement with submarines of category j, $x_{ij}$, equals the product of four terms:

1. The initial number of category j submarines committed to the battle, $Y_{oj}$.
2. The expected number of patrols made by a category j submarine during the battle, $P_j$.
3. The expected number of engagements per submarine of category j with merchant ships of category i during one patrol, $N_{ij}$.
4. The expected number of merchant ships of category i lost per engagement with submarines of category j, $M_{ij}$.

i.e.

$$x_{ij} = Y_{oj} P_j N_{ij} M_{ij} . \qquad \text{[Eq. 1]}$$

The expected number of <u>escorts</u> of category i <u>lost</u> in a battle of duration t as a result of engagements with submarines of category j, $z_{ij}$, equals the product of terms 1, 2 and 3 and the expected number of <u>escorts</u> of category i lost per engagement with submarines of category j, $E_{ij}$, i.e.

$$z_{ij} = Y_{oj} P_j N_{ij} E_{ij} . \qquad \text{[Eq. 2]}$$

The expected number of <u>submarines</u> of category j lost in a battle of duration t is the product of the initial number of category j submarine, $Y_{oj}$, and the proportion of these submarines that does not survive both the direct and indirect AS defences, $F_j$, i.e.

$$y_j = Y_{oj} F_j . \qquad \text{[Eq. 3]}$$

The terms $N_{ij}$, $P_j$ and $F_j$ defined above depend on more basic quantities such as the duration of the campaign, t, the submarine cycle time, $T_{cj}$, the A/S barrier attrition, $B_j$, the probability that a submarine is lost in an engagement, $K_{ij}$, the submarine endurance-limited time on station, $T_{sj}$, the time required by a submarine to complete an engagement, $\tau$, the engagement rate per disengaged submarine, $\lambda_{ij}$, and the maximum number of engagements per submarine patrol, $\ell_{ij}$. The inter-relationship between these quantities and the merchant ship, escort and submarine losses is indicated in Figs. 1 and 2. The boxes in these figures represent models that transform box inputs into box outputs according to the equations referred to in each box. These equations are derived in the appendices.

Most of the inputs listed on the left of Figs. 1 and 2 depend on more basic quantities. For example, the engagement rate per submarine depends on the flow rate of shipping through the submarines' operating area, the size of this area, the range at which the submarine can detect shipping, the closure speed and tactics of the submarine, and the number of false targets pursued by the submarine. Submodels have been developed for estimating these quantities and are discussed in Appendix C.

## 3. ANALYSIS OF LOSSES

In making a broad comparison of the results of two or more defence-of-shipping campaigns it is generally enlightening to compare the overall losses of merchant ships, submarines, and escorts. These losses are easily obtained by summing the losses given by Eqs. 1, 2 and 3 over the I categories of merchant ships and the J categories of submarines.

In analysing the results of a particular campaign, one can resolve the overall losses into components and factors that illuminate a number of important questions. This can be seen by referring to Fig. 3 which, for simplicity, applies to a situation where only two categories of submarines (nuclear submarines deployed in the Western and Eastern anti-shipping areas) and only merchant-ship losses are considered.

Resolving the overall losses into the component losses due to each category of submarine indicates where the losses occur. This is indicated in the second row of circles in Fig. 3. Expressing each of these component losses as the product of the losses per submarine, $m_{\cdot j}$, and the corresponding initial number of submarines, $Y_{0j}$, and then comparing the losses per submarine (indicated by the fourth row of circles in Fig. 3) reveals which submarines are most effective. Factoring the losses per submarine into the product of the number of patrols, the number of engagements per patrol and the number of ships lost per engagement and reviewing those three quantities indicates why some categories of submarines are more effective than others. This is indicated by the fifth row of circles in Fig. 3.

More specific reasons for differences in the effectiveness of different categories of submarines can be traced to differences in

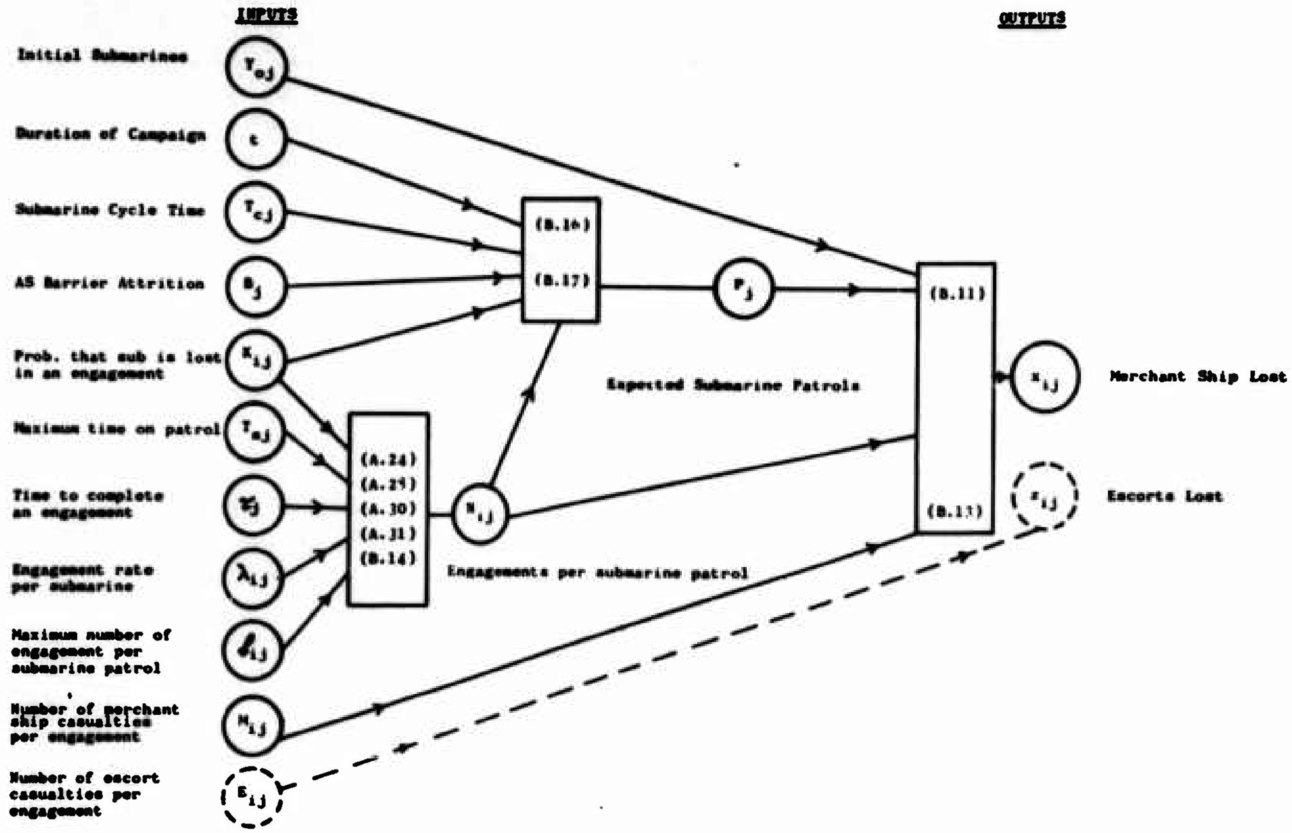


FIG. 1 INTER RELATIONSHIP OF QUANTITIES USED IN MODELS FOR MERCHANT SHIP AND ESCORT LOSSES

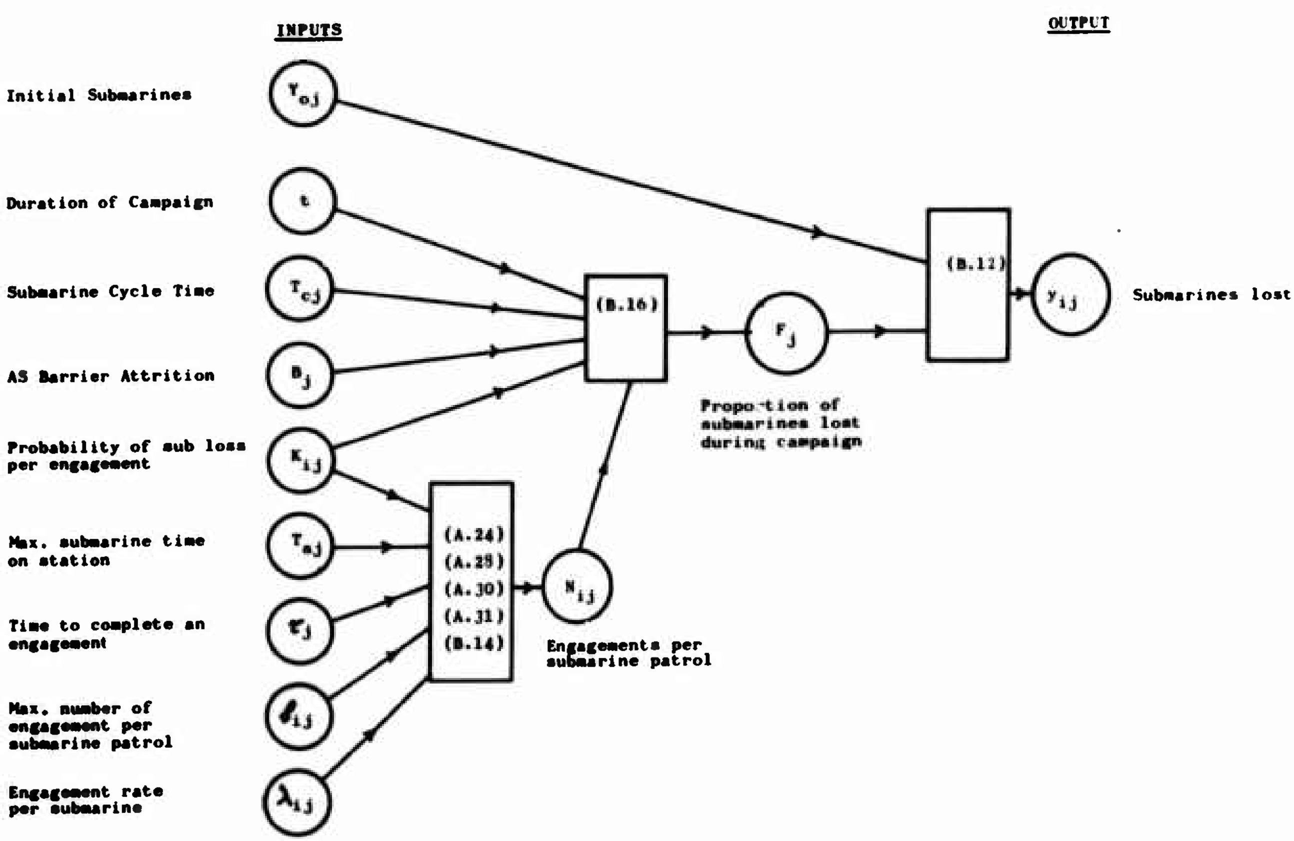


FIG. 2 INTER-RELATIONSHIP OF QUANTITIES USED IN MODEL FOR SUBMARINE LOSSES

more basic quantities such as the inputs of Fig. 1 and quantities from which they were derived. As an example, suppose that the submarines deployed in the Western area are more effective than those deployed in the Eastern area because they have a larger expected number of patrols. Tracing this difference through the relationship of Fig. 1 one can determine why this is so. A smaller submarine cycle time or A/S barrier attrition for the submarine in the threat, for instance, might be the reason for the difference.

More detailed submodels, such as one describing the A/S barrier attrition, can shed additional light on the reasons for the differences in the effectiveness of submarines of different categories.

The rationale underlying Fig. 3 can be applied to realistic scenarios involving dozens of categories of submarine-operating areas. It can also be oriented to trace losses according to ship categories and shipping routes, and can treat submarine losses and escort losses in addition to the merchant ship losses. The basis of the rationale is that overall losses can be viewed as the output of a hierarchy of models, and by tracing component losses through this hierarchy, one gains insight concerning where and why losses differ. Such information is useful not only for understanding the results of a given defence-of-shipping campaign, but also for comparing the results of one campaign with another. Furthermore, the process of tracing the reasons for losses back to more fundamental quantities give the analyst a precise understanding of how certain assumptions influence campaign results.

## 4. SENSITIVITY ANALYSES

### 4.1 Purposes

In addition to being useful for analysing the losses resulting from a defence-of-shipping campaign, the hierarchy of models represented in Fig. 3 can be used to perform sensitivity analyses at various levels in the hierarchy. At the higher level in the hierarchy, one can study the effect of varying the initial deployment of submarines, $Y_{oj}$, and the expected ship losses per submarine, $m_{.j}$, and the overall losses, $x_{..}$. Below this level, one can study the effect of changing the expected number of patrols per submarine, $P_j$, expected number of engagements per patrol, $N_{.j}$, and the expected number of ships lost per engagement, $\bar{M}_j$, on the losses per submarine, $m_{.j}$, and on the overall losses, $x_{..}$. Proceeding to the next level down in the hierarchy one can study the effect of changing the input to Fig. 1 on any of the following quantities:

a. The number of patrols per submarine, $P_j$; the number of engagements per patrol, $N_{.j}$; the number of ships lost per engagement, $\bar{M}_j$.

b. The expected number of ships lost per initial submarine, $m_{.j}$.

c. The overall ship losses, $x_{..}$.

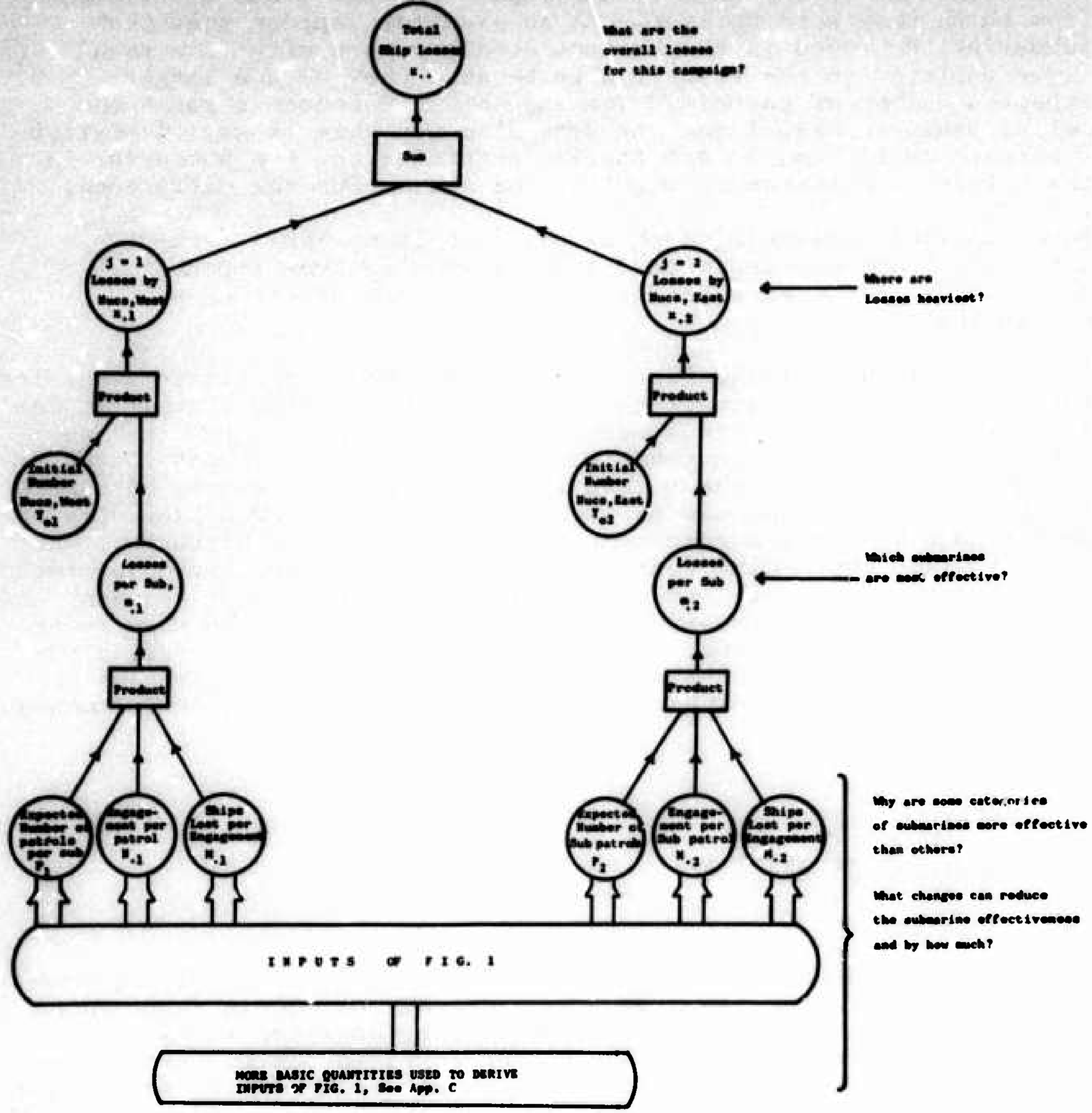


FIG. 3 HIERARCHY OF MODELS LEADING TO SHIP LOSSES

below this level, one can study the effect of changing more basic quantities on any of the related quantities above it in the hierarchy of models. As an example, suppose one were interested in estimating the effect on overall ship losses of increasing the range at which a certain class of submarine can detect shipping. The increased detection range increases the engagement rate according to the equation of Appendix C. In turn this increase in engagement rate increases the number of engagements per patrol, $N_{\cdot j}$, and changes the number of patrols per submarine, $P_j$, according to the relationship in Fig. 1. These changes in term increase the number of ships lost per submarine, $m_{\cdot j}$, and the total number of ships loss, $x_{\cdot\cdot}$.

## 4.2 Remarks Concerning Measures of Effectiveness for Tactics

Many analyses, focusing on lower-level relationships, such as those associated with tactics, have traditionally selected lower-level measures of effectiveness, which may not reflect the overall effectiveness of a defence-of-shipping campaign. A hierarchy of models such as that represented in Fig. 1 can be used to assess the validity of a particular choice of measure of effectiveness at the tactical level by studying the sensitivity of an overall measure of effectiveness, such as total ship losses, to a number of lower level quantities, including the tactical level measure of effectiveness.

For example, a common measure of effectiveness used for evaluating alternate A/S tactics is the probability of killing a submarine in an engagement, denoted by $K_{ij}$ in Fig. 1. From Fig. 1 it is clear that the A/S tactic with the largest value of $K_{ij}$ in general does not minimize the overall number of ships lost because there are several ways of reducing ship losses that do not depend on $K_{ij}$.

One way is to reduce the engagement rate per submarine, $\lambda_{ij}$. This can be accomplished by deception, by degrading the submarines detection capability, by rerouting the shipping, etc. Another way of reducing ship losses is to increase the submarine's cycle time, $T_{cj}$. This could be achieved by enticing submarines to deploy in more remote areas, or by forcing them to use more distant bases. Still another way of reducing ship losses without changing $K_{ij}$ is to increase the A/S barrier attrition, $B_j$. This can be done either by increasing the level of A/S forces assigned to the barrier, or by luring more submarines through the barrier.

The relationships represented by Figs. 1 and 2 provide a mean for making quantitative comparisons of these various ways of reducing ship losses. They also provide a link between measures of effectiveness traditionally used at the tactical level and those used at the campaign level. Determination of such links for the more likely kinds of defence-of-shipping campaigns should be of considerable value in evaluating alternate tactics.

## CONCLUSION

Models have been developed for the losses of merchant ships, escorts, and submarines resulting from large scale ASW defence-of-shipping campaigns involving both direct and indirect forms of protection.

These models are consistent with statistic models of Refs. 2 and 3, yet sufficiently simple to permit large scale campaigns, involving many different categories of merchant ships and submarines operating in different geographical areas, to be treated.

The loss expressions, which are closed-form solutions of differential equations, can be factored into terms that have simple interpretation. This factorization is the basis for a hierarchy of models that leads from basic quantities, such as submarine detection range, cycle time, endurance, weapon load, etc., through a number of steps to the overall expected losses of merchant ships, escorts and submarines. The hierarchy of models provides a convenient means of understanding why losses differ among the various categories of merchant ships, escorts, submarines, and geographical areas. The hierarchy of models is also useful for conducting sensitivity analyses and appreciating how measures of effectiveness at the tactical level are related to those at the campaign level.

# APPENDICES and REFERENCES

## APPENDIX A

## THE EXPECTED NUMBER OF DISENGAGED SUBMARINES ON PATROL AND RELATED QUANTITIES

### A.1 Introductory

In this appendix a number of concepts and expressions are developed that lead to an expression for the expected number of disengaged submarines on patrol. These are:

1. The expected number of submarines on patrol, $S_p$.
2. The expected time on patrol in one submarine cycle, $T_p$.
3. Engaged and disengaged time on patrol.
4. The number of engagements per patrol and related quantities.
5. Expressions for number of engagements and related quantities when several targets are present.
6. Summary of main results.
7. Remarks concerning submarines on patrol and disengaged submarines on patrol.

These concepts have been introduced to account for the non-zero duration of submarine attacks and different types of targets (e.g. convoys and independents), and to overcome some of the computational difficulties associated with employing the stochastic models of Ref. 2.

### A.2 The Expected Number of Submarines on Patrol, $S_p$

The expected number of submarines on patrol depends on the rate at which submarines enter the antishipping area and on how long they stay there. If $\mu(\tau)$ is the rate at which submarines enter the antishipping area at time $\tau$, and $h(t, \tau)$ is the probability that a submarine will remain on patrol until time $t$, given that it arrive at time $\tau$, then the expected number of submarines is given by:

$$S_p(t) = \int_{t-T_m}^{t} \mu(\tau)\, h(t, \tau)\, d\tau , \qquad \text{[Eq. A.1]}$$

where $T_M$ is the maximum time a submarine can spend in the patrol area in one submarine cycle. Applying the mean value theorem to Eq. A.1 yields:

$$S_p(t) = \mu(\xi)\, \bar{T}_p(t), \quad t - T_m \le \xi \le t, \qquad \text{[Eq. A.2]}$$

where

$$\bar{T}_p(t) = \int_{t-T_s}^{t} h(t, \tau)\, d\tau . \qquad \text{[Eq. A.3]}$$

The rate at which submarines enter the antishipping patrol area, $\mu$, can be approximated as follows:

$$\mu(\xi) = \frac{Y(\xi)}{T_c}, \qquad \text{[Eq. A.4]}$$

where $T_c$ is the cycle time of a submarine.

Equation A.4 may be viewed either as the value of the submarine entry rate smoothed over a submarine cycle, or as the submarine entry rate associated with the submarines being uniformly distributed over their normal operating cycles. Substitution of Eq. A.4 into Eq. A.2 yields:

$$S_p(t) = \frac{\bar{T}_p\, Y(t)}{T_c} . \qquad \text{[Eq. A.5]}$$

## A.3 The Expected Time on Patrol in One Submarine Cycle, $\bar{T}_p$

Since the probability of a submarine being destroyed in a patrol area depends on the number of engagements, and since the engagement rate is assumed to be constant, the probability that a submarine survives from time $\tau$ to time $t$ depends on its duration of exposure $t - \tau$, and otherwise does not depend on $t$ or $\tau$. Therefore:

$$h(t, \tau) = h(v), \qquad \text{[Eq. A.6]}$$

where

$$v = t - \tau .$$

Substitution of Eq. A.6 into Eq. A.3 yields:

$$\bar{T}_p = \int_0^{T_m} h(v)\, dv . \qquad \text{[Eq. A.7]}$$

The integrand of Eq. A.7, $h(v)dv$, can be interpreted as the increase in the expected time on patrol that occurs during the time interval between $v$ and $v+dv$. The reason for this is that $dv$ is the increase in time given that the submarine is still on patrol at time $v$, and $h(v)$ is the probability that it is still on patrol at that time. Consequently, $T_p$ may be interpreted as the expected time on patrol in one submarine cycle.

## A.4 Engaged and Disengaged Time on Patrol

Suppose that a submarine can have at most two engagements on a patrol, and that the starting times and durations of these engagements are specified. Under these conditions, its surviving probability curve would have the behaviour shown in Fig. A.1.

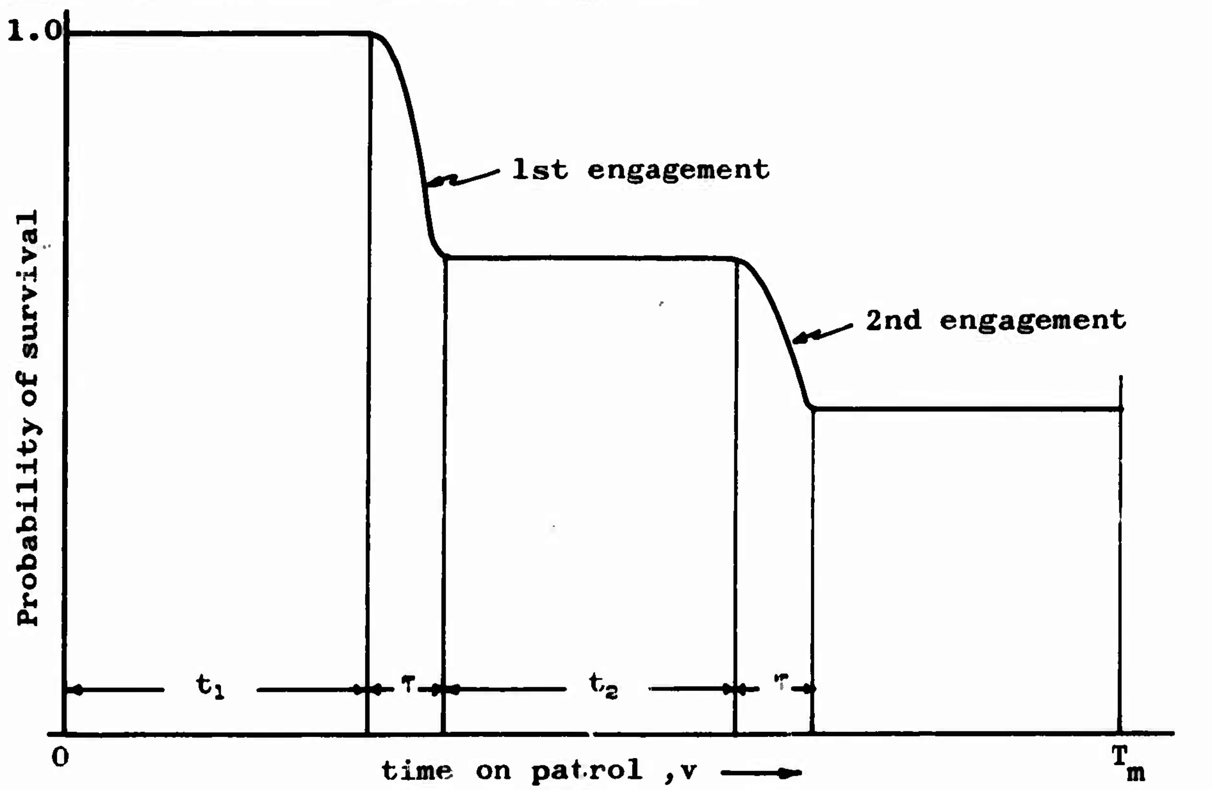


FIG. A.1 SUBMARINE SURVIVAL PROBABILITY vs TIME ON PATROL

Since a submarine on patrol has already passed through indirect defences, it may be lost only as a result of engagements. Consequently the survival probability of Fig. A.1 drops only during engagements. During the periods between engagements the survival probability is constant.

In this situation, if the time on patrol is partitioned into two parts:

1. the engaged time on patrol, and
2. the disengaged time on patrol,

then all losses may be considered to occur during the engagement time, and all new engagement may be considered to be generated during the disengaged time.

Furthermore, if $T_m$ is the maximum time on patrol of the submarine, and the submarine has N engagements, each requiring a time $\tau$ to complete, then the maximum disengaged time per patrol is given by:

$$T_s^* = T_m - N\tau \; . \qquad \text{[Eq. A.8]}$$

Using the above notions, the most significant part of Fig. A.1 can be replaced by Fig. A.2 where engagements are assumed to occur instantaneously, and the disengaged time on patrol, u, varies from zero to a maximum value of $T_s^*$.

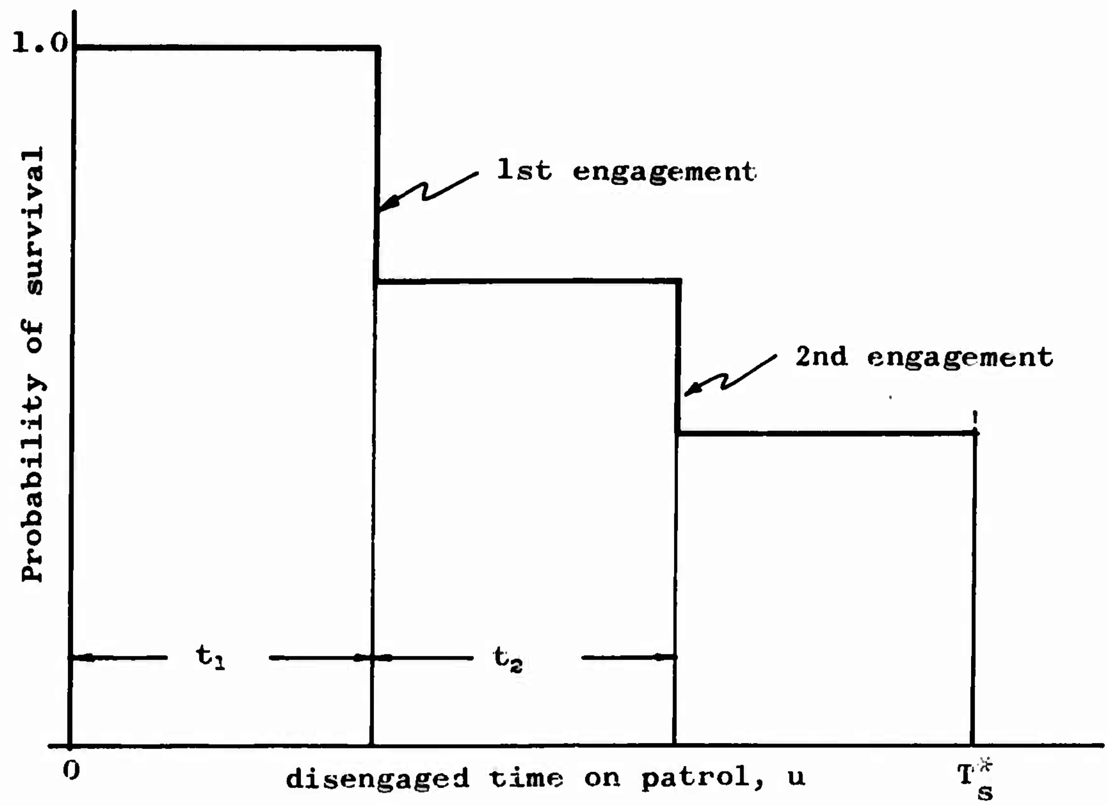


FIG. A.2 SUBMARINE SURVIVAL PROBABILITY vs DISENGAGED TIME ON PATROL

It should be noted that survival probability curves are generally smoother than indicated in Figs. A.1 and A.2. The reason for this is that the precise times at which engagements start are random and in general could occur almost anywhere between 0 and $T_s^*$.

## A.5 The Number of Engagements per Patrol and Related Quantities

In this section the number of engagements is related to the expected time on patrol, the expected disengaged time on patrol, and corresponding engagement rates.

The number of engagements per patrol, N, depends on the following quantities:

a. The engagement rate per disengaged submarine, $\lambda$.

b. The endurance-limited time on station, $T_m$.

c. The time required to complete an engagement, $\tau$.

d. The maximum number of attacks (derivable from the torpedo load), $\ell$.

e. The probability that the submarine is lost in an engagement, k.

Before accounting for all of these factors, it is convenient to consider the situation where submarines are endurance limited. This amounts to ignoring the last two factors, which are accounted for later.

### Endurance-Limited Submarines

For a submarine which is endurance limited, i.e. it neither runs out of torpedoes nor is sunk before the end of a patrol, the expected number of engagements per patrol is

$$\tilde{N} = \lambda T_s^* , \qquad \text{[Eq. A.9]}$$

where $T_s^*$ is the maximum disengaged time per patrol.

If N in Eq. A.8 is also interpreted as the expected number of engagements per patrol in the endurance-limited case, then Eqs. A.8 and A.9 give:

$$T_s^* = \frac{T_m}{1 + \lambda\tau}$$

and

$$N = \frac{\lambda}{1 + \lambda\tau} \cdot T_m \equiv \lambda' \; T_m \; . \qquad \text{[Eq. A.10]}$$

Equation A.10 suggests that the number of engagements per patrol can be estimated in terms of total (max.) time on patrol, $T_m$, (as opposed to disengaged time on patrol, $T_s^*$) if the engagement rate is appropriately decreased.

For this alternate viewpoint the time between engagements includes both the disengaged and the engaged time. This is clear from the following equation for the mean time between engagements:

$$\frac{1}{\lambda'} = \frac{1}{\lambda} + \tau \; . \qquad \text{[Eq. A.11]}$$

The term $1/\lambda$ is the mean time required by the disengaged submarine to find a new target, and $\tau$ is the mean time required by an engaged submarine to complete an engagement.

Since it is more convenient to work in total time rather than disengaged time, the second viewpoint involving total time and engagement rate $\lambda^1$, will be emphasized in the following development. Mean disengaged time can be obtained from mean total time by multiplying the total time by $\frac{\lambda^1}{\lambda}$, e.g. the expected disengaged time per patrol is given by:

$$\bar{T}_s^* = \frac{\lambda^1}{\lambda} \bar{T}_p \, . \qquad \text{[Eq. A.12]}$$

## The Effect of Torpedo Load and Vulnerability

The value of N given by Eq. A.10 over-estimates the number of engagements because neither the finite weapon load nor the vulnerability of the submarine were taken into account. This can be done in a number of ways, depending on what assumptions are made regarding the engagement generating process. Here two cases are considered.

## Case 1. Poisson Process with Intensity, $\lambda^1$

A submarine which is able to make at most $\ell$ attacks will remain on patrol only if it has made fewer than $\ell$ attacks and survived all of them. Mathematically this can be stated as follows:

$$h(v) = \sum_{n=0}^{\ell-1} d_n(v)\,(1-K)^n, \quad v < T_s \, , \qquad \text{[Eq. A.13]}$$

where $d_n(v)$ = probability that the submarine has n complete engagements in time v, given that it survives at time v.

$1-K$ = probability that a submarine survives an engagement [and so $(1-K)^n$ is the probability of surviving n engagements].

Assuming the engagements occur according to a Poisson process with intensity $\lambda^1$, it follows that:

$$d_n(v) = \frac{(\lambda^1 v)^n}{h!} \; e^{-\lambda^1 v} \, , \; v < T_s \, . \qquad \text{[Eq. A.14]}$$

Combining Eqs. A.7 and A.14 yields for the mean time on patrol:

$$\bar{T}_p = \sum_{n=0}^{\ell-1} \frac{(1-K)^n}{\lambda^1 \; n!} \; I_n(\lambda^1 \, T_m) \, , \qquad \text{[Eq. A.15]}$$

where

$$I_n(\lambda^1 T_m) = \int_0^{\lambda^1 T_m} x^n e^{-x} dx, \quad n = 0, 1, \ldots \qquad \text{[Eq. A.16]}$$

the integrals $I_n(\lambda^1 T_m)$ can be evaluated using the equations

$$I_0(\lambda^1 T_m) = (1 - e^{-\lambda^1 T_m}) \qquad \text{[Eq. A.17]}$$

and

$$I_n(\lambda^1 T_m) = n I_{n-1}(\lambda^1 T_m) - (\lambda^1 T_m)^n e^{-\lambda^1 T_m}, \quad n = 1, 2, \ldots \qquad \text{[Eq. A.18]}$$

Rewriting Eq. A.15 to highlight the number of engagements gives

$$N = \sum_{n=0}^{\ell-1} \frac{(1-K)^n}{n!} I_n(\tilde{N}), \qquad \text{[Eq. A.19]}$$

where

$$\tilde{N} = \lambda^1 T_m \quad \text{and} \quad N = \lambda^1 \bar{T}_p .$$

According to Eq. A.10, N may be interpreted as the number of engagements that a submarine should make during one patrol period if it is not torpedo limited and not killed during the patrol. N is the mean number of complete engagements during the patrol including torpedo limiting and kill probability in an engagement. (It is easy to see from Eq. A.6 that in Eq. A.19 when $K = 0$ and $\ell = \infty$: $N = \tilde{N}$).

## Case 2. Continuous Engagements

If the discrete nature of the engagement process is ignored, then it may be assumed that precisely $\lambda^1 t$ engagements occur in time $t$ provided the submarine survives that long and does not run out of torpedoes. Although this assumption is less realistic than the preceding one, it yields simple results which may be sufficiently accurate for some purposes, and provides a simple means of estimating whether submarines are torpedo limited.

First consider what happens when the submarine survives all engagements but runs out of torpedoes. The potential number, $\lambda^1 T_m$, of engagements that would be realized over the entire patrol period exceeds $\ell$ and in this case $\ell$ is the number of engagements.

More generally, if torpedo load is taken into account, but submarine vulnerability is not, then the number of engagements per patrol is given by

$$N^* = \min(\lambda^1 T_m, \ell),$$

and the corresponding time required to make these engagements is

$$T^* = \frac{N^*}{\lambda^1} .$$

Now consider what happens when submarine vulnerability is taken into account. In an increment of time $\Delta t$, the probability that the submarine will have an engagement is $\lambda^1 \Delta t$, and the probability that the submarine is sunk as a result of an engagement is $\lambda^1 K \Delta t$. Under these conditions it follows that the probability that a submarine survives from time zero to time $v$ and has not run out of torpedoes, is:

$$h(v) = \begin{cases} e^{-\lambda^1 K v} , & v \leq T^* , \\ 0 , & v > T^* . \end{cases}$$

From Eq. A.7 it follows that the expected time on patrol in one submarine cycle is:

$$\bar{T}_p = \frac{1}{\lambda^1 K} (1 - e^{-\lambda^1 T^* K}) , \qquad \text{[Eq. A.20]}$$

and the corresponding number of engagements per patrol is

$$N = \bar{T}_p \lambda^1 = \frac{1}{K} (1 - e^{-N^* K}) .$$

## A.6 Expressions for Number of Engagements and Related Quantities when Targets of Several Different Classes are Present

When several different target classes are present, the parameters $\lambda$, $K$, $\ell$ and $\tau$ will generally be affected. The parameter $T_m$ should not be affected because it depends primarily on characteristics of the submarine, such as the food supply, and may be assumed to be independent of the target mix.

The following subsections provide expressions for estimating the parameters $\lambda$, $K$, $\ell$ and $\tau$ for a single submarine class and several target classes.

### A.6.1 Engagement Rate, $\lambda_{\cdot j}$

The total engagement rate between a submarine of class $j$ and all targets, $\lambda_{\cdot j}$, is the sum of the engagement rates for all of the target classes:

$$\lambda_{\cdot j} = \sum_{i=1}^{I} \lambda_{ij} . \qquad \text{[Eq. A.21]}$$

The probability that a submarine of class j has an engagement with a target of class i, given that it has an engagement, $P_{ij}$, is the relative frequency of occurrence of such encounter, so that:

$$P_{ij} = \frac{\lambda_{ij}}{\lambda_{\cdot j}} . \qquad \text{[Eq. A.22]}$$

### A.6.2 The Probability that a Submarine of Class j is Lost in an Engagement with Targets from Several Different Classes, $\bar{K}_j$

Suppose that a submarine of class j has an engagement. It will engage a target of class i with probability $P_{ij}$, in which case it will be lost with probability $K_{ij}$; the expected probability that it will be lost as a result of an engagement with targets from I different classes is:

$$\bar{K}_j = \sum_{i=1}^{I} P_{ij} K_{ij} , \qquad \text{[Eq. A.23]}$$

or, in view of Eq. A.2,

$$\bar{K}_j = \frac{\sum_{i=1}^{I} \lambda_{ij} K_{ij}}{\lambda_{\cdot j}} . \qquad \text{[Eq. A.24]}$$

### A.6.3 The Expected Time to Complete an Engagement

By reasoning analogous to the preceding paragraph, the expected time for a submarine of class j to complete an engagement with targets from several classes is

$$\bar{\tau}_j = \frac{\sum_{i=1}^{I} \lambda_{ij} \tau_{ij}}{\lambda_{\cdot j}} , \qquad \text{[Eq. A.25]}$$

where $\tau_{ij}$ is the expected time for a submarine of class j to complete an engagement with a submarine of class i.

The engagement rate $\lambda^1$ will be given by:

$$\lambda^1_j = \frac{\lambda_{\cdot j}}{1 + \lambda_{\cdot j} \bar{\tau}_j} . \qquad \text{[Eq. A.26]}$$

### A.6.4 The Number of Engagements Required by a Submarine of Class j to Expend all of its Weapons, $\ell_j$

Let $W_j$ be the weapon load of a submarine of class j and let $w_{ij}$ be the average $\bar{\tau}$ number of weapons fired by such a submarine in each engagement with target of class i. Then the average number of engagements required to exhaust the weapon load <u>if only class i targets were present is</u>:

$$\ell_{ij} = \frac{W_j}{w_{ij}} . \qquad \text{[Eq. A.27]}$$

Equation A.27 defines $\ell_{ij}$. The average number of torpedoes spent in an engagement is:

$$\frac{\sum_{i=1}^{I} \lambda_{ij} w_{ij}}{\lambda_{\cdot j}} ,$$

The average number of attacks per submarine patrol $\bar{\ell}_j$ can be obtained by dividing the total weapon load by this average number:

$$\bar{\ell}_j = \frac{W_j}{\sum_{i=1}^{I} \frac{\lambda_{ij} w_{ij}}{\lambda_{\cdot j}}} = \frac{\lambda_{\cdot j}}{\sum_{i=1}^{I} \frac{\lambda_{ij}}{\ell_{ij}}} . \qquad \text{[Eq. A.28]}$$

An equivalent form of Eq. A.28 based on Eq. A.22 is

$$\frac{1}{\bar{\ell}_j} = \sum_{i=1}^{I} P_{ij} \frac{1}{\ell_{ij}} . \qquad \text{[Eq. A.29]}$$

In other words, $\bar{\ell}_j$ is the weighted harmonic mean of $\ell_{ij}$'s with weights $P_{ij}$ given by Eq. A.22.

## A.7 Summary of Main Results

When several target classes are present the following expressions, which are extensions of Eqs. A.15, A.12, A.5, A.19 and A.20, may be used:

a. The expected number on patrol in one submarine cycle for a submarine of class j :

$$\bar{T}_{pj} = \sum_{n=0}^{\bar{\ell}_j - 1} \frac{(1 - \bar{K}_j)^n}{\lambda_j^1 \, n!} I_n(\lambda_j^1 T_m) , \qquad \text{[Eq. A.30]}$$

where from Eqs. A.11 and A.26

$$\lambda_j^1 = \frac{1}{1 + \lambda_{\cdot j} \bar{\tau}_j} \lambda_{\cdot j} ,$$

and Eq. A.16 gives $I_n(\lambda_j^1 T_{mj})$.

b. The expected disengaged time in one submarine cycle for a submarine of class $j$:

$$\bar{T}_{sj} = \frac{\lambda^1_j}{\lambda_{.j}} \bar{T}_{pj} \,. \qquad \text{[Eq. A.31]}$$

c. The expected number of disengaged submarine of class $j$ on patrol:

$$S_j = \frac{\bar{T}_{sj}}{T_{cj}} Y_j \,, \qquad \text{[Eq. A.32]}$$

where $T_{cj}$ is the cycle time of a class $j$ submarine ($T_{cj} = \bar{T}_{pj}$ + time off station per patrol).

d. The expected number of engagements per submarine of class $j$ per patrol:

$$N_j = \lambda^1_j \bar{T}_{pj} = \lambda_{.j} \bar{T}_{sj} \,. \qquad \text{[Eq. A.33]}$$

e. A simplified expression for $\bar{T}_{pj}$, from Eq. A.20, is:

$$\bar{T}_{pj} = \frac{1}{\lambda^1_j \bar{K}_j} \left\{ 1 - \exp \bar{K}_j \left[ \min \left( \ell_j, \lambda^1_j T_s \right) \right] \right\} . \qquad \text{[Eq. A.34]}$$

## A.8 Remarks Concerning Submarines on Patrol and Disengaged Submarines on Patrol

In the preceding development, engagements may be considered to be generated in two ways: be disengaged submarines on patrol at a rate of $\lambda_{.j}$ per submarine or by all submarines on patrol at a reduced rate of $\lambda^1_j$ per submarine. The corresponding engagement rates, taking the number of submarines into account, are

$\lambda_{.j} S_j$ and $\lambda^1_j S_{pj}$,

where $S_{pj}$ is the number of submarines of class $j$ on patrol (engaged plus disengaged). These engagement rates are equivalent, as can be seen from Eqs. A.10, A.12 and A.19. Consequently, Lanchester attrition equations may be formulated either in terms of $\lambda_{.j} S_j$ or $\lambda^1_j S_{pj}$.

## APPENDIX B

## FORMULATION AND SOLUTION OF THE LANCHESTER DIFFERENTIAL EQUATIONS

### B.1 Formulation

In Ref. 1 a stochastic model was developed for the losses of convoyed merchant ships and submarines. According to this model the expected number of merchant ships lost, (see Eqs. 16 and 22 of Ref. 1) can be shown to reduce the following equation:

$$\frac{dx}{dt} = \lambda(t)\ S_p(t)\ M \qquad \text{[Eq. B.1]}$$

where

$x$ = the expected number of merchant ships lost

$t$ = time

$\lambda(t)$ = the engagement rate per submarine on patrol

$S_p(t)$ = the number of submarines on patrol; see Eq. A.5

$M$ = the expected number of merchant ships lost per engagement.

In contrast with this memorandum, Ref. 1 emphasizes attacks rather than engagements; also the notation differs. Table B.1 is helpful for translating between Eq. B.1 and the results of Ref. 2.

TABLE B.1

CORRESPONDENCE OF SYMBOLS

| This Memorandum | Reference 2 |
|---|---|
| $\tau(t)$ | $\lambda(t)$ |
| $h(\tau, t)$ | $b(\tau, t)$ |
| $\mu(\tau)$ | $\mu(\tau)[1-p_1(\tau)]$ |
| $M$ | $P_1(t)\ E(m)$ |

Although Eq. B.1 has its roots in a stochastic analysis of Ref. 1, it can be derived intuitively by observing that the expected loss rate, $\frac{dx}{dt}$, equals the product of the engagement rate per submarine on patrol, $\lambda(t)$, the number of submarines on patrol, $S_p$, and the expected number of merchant ships lost per engagement.

The analyses of Refs. 1, 2 and 3 considered only a single category of targets (convoys) and assumed that the duration of an attack was zero. These shortcomings were overcome in Ref. 5 where the stochastic model was extended to cover the first factors. The second one is overcome in this memorandum by considering only the disengaged submarines on patrol (because submarines are engaged during an attack), and considering average targets whose properties represent the actual mixture of different target types. See Appendix A for details.

In view of these considerations and the fact that there are I classes of merchant ships and J classes of submarines to be considered, a more general Lanchester equation may be written as follows:

$$\frac{dx_{ij}}{dt} = \lambda_{ij} S_j M_{ij} \qquad \begin{matrix} i = 1, 2, \ldots I \\ j = 1, 2, \ldots J \end{matrix} \qquad \text{[Eq. B.2]}$$

where

$x_{ij}$ = the expected number of merchant ships of category i lost to submarines of category j in time t

$\lambda_{ij}$ = the expected engagement rate of a category j submarine with targets of category i

$S_j$ = the expected number of disengaged category j submarines on patrol

$M_{ij}$ = the expected number of merchant ships of category i lost per engagement with a submarine of category j.

Since escorts accompanying merchant ships of category i may also be lost as a result of engagements with submarines of category j, the loss rate equation for these escorts, by analogy, is:

$$\frac{dz_{ij}}{dt} = \lambda_{ij} S_j E_{ij} \qquad \begin{matrix} i = 1, 2, \ldots I \\ j = 1, 2, \ldots J \end{matrix} \qquad \text{[Eq. B.3]}$$

where

$z_{ij}$ = the expected number of escorts with merchant ships of category i that are lost as a result of engagements with submarines of category j up to time t

$E_{ij}$ = the expected number of escorts with category i merchant ships sunk per engagement with a category j submarine.

Submarines may be lost as a result of exposure to either the direct or indirect defences. The loss rate of category j submarines due to the direct defences surrounding category i merchant ships by the above argument is $\lambda_{ij} S_j K_{ij}$, where $K_{ij}$ is the probability that a category j submarine is lost in an engagement with category i merchant ships. The loss rate of category j submarines caused by indirect defences, smoothed over one submarine cycle, is approximately* equal to $B_j Y_j / T_{cj}$, where $B_j$ is the probability that a category j submarine is destroyed by the indirect defensive forces in one submarine cycle, $T_{cj}$ is the category j submarine cycle time, and $Y_j$ is the number of surviving submarines of category j. The loss-rate equation for category j submarines therefore is:

$$\frac{dy_j}{dt} = \sum_{i=1}^{I} \lambda_{ij} \, S_j \, K_{ij} + \frac{B_j}{T_{cj}} Y_j, \quad j = 1, 2, \ldots J \qquad \text{[Eq. B.4]}$$

where $y_j$ is the expected number of category j submarine lost in time t.

## B.2 Solution

Assuming that no new submarines are added to the battle after the war starts, the expected number of submarines lost at time t, $y_j$, equals the initial number, $Y_{oj}$, minus the number surviving at time t, $Y_j$. Consequently,

$$\frac{dy_j}{dt} = - \frac{dY_j}{dt} \ . \qquad \text{[Eq. B.5]}$$

Substitution of Eq. B.4 and Eq. A.32 into Eq. B.3 and integrating yields:

$$Y_j = Y_{oj} \exp(-a_j t) \qquad \text{[Eq. B.6]}$$

where

$$a_j = \frac{B_j + \bar{T}_{sj} \sum_{i=1}^{I} \lambda_{ij} \, K_{ij}}{T_{cj}} \qquad \text{[Eq. B.7]}$$

so that

$$y_j \equiv Y_{oj} - Y_j = Y_{oj}[1 - \exp(-a_j t)] \ . \qquad \text{[Eq. B.8]}$$

Substitution of Eq. A.32 into Eq.B.2 and Eq. B.3 and integration yields:

$$x_{ij} = \frac{\lambda_{ij} \, \bar{T}_{sj} \, M_{ij}}{B_j + \bar{T}_{sj} \sum_{i=1}^{I} \lambda_{ij} \, K_{ij}} \, y_j \qquad \text{[Eq. B.9]}$$

$$z_{ij} = \frac{\lambda_{ij} \, \bar{T}_{sj} \, E_{ij}}{B_j + \bar{T}_{sj} \sum_{i=1}^{I} \lambda_{ij} \, K_{ij}} \, y_j \ . \qquad \text{[Eq. B.10]}$$

Expressions for $\bar{T}_{sj}$, which appear in Eqs. B.7, B.9 and B.10, are derived in Appendix A where it is shown that $\bar{T}_s$ depends on

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*This simplifying assumption is not as refined as that used in Ref. 1 and leads to a simpler expected value equation.

aggregated properties of targets in the I different classes. In particular it is shown that the aggregated values of the probability that a submarine of class j is lost in an engagement is

$$\bar{K}_j = \frac{\sum_{i=1}^{I} \lambda_{ij} K_{ij}}{\lambda_{.j}} . \qquad \text{[Eq. B.11]}$$

Substituting Eq. B.11 into B.7, B.9 and B.10 yields the following results:

$$x_{ij} = N_{ij} P_j M_{ij} Y_{oj} , \qquad \text{[Eq. B.12]}$$

$$y_j = F_j Y_{oj} , \qquad \text{[Eq. B.13]}$$

$$z_{ij} = N_{ij} P_j E_{ij} Y_{oj} , \qquad \text{[Eq. B.14]}$$

where

$$N_{ij} = \lambda_{ij} \bar{T}_{sj} \qquad \text{[Eq. B.15]}$$

$$N_{.j} = \lambda_{.j} \bar{T}_{sj} \qquad \text{[Eq. B.16]}$$

$$F_j = 1 - \exp\left[ - \left( \frac{B_j + N_{.j} \bar{K}_j}{T_{cj}} \right) t \right] \qquad \text{[Eq. B.17]}$$

$$P_j = \frac{F_j}{B_j + N_{.j} \bar{K}_j} . \qquad \text{[Eq. B.18]}$$

The interpretations of $N_{ij}$ and $N_{.j}$ follow directly from Eqs. B.15 and B.16. The expected number of engagements by a submarine in one patrol equals the product of the engagement rate and the time that the submarine is disengaged during one patrol, $\bar{T}_{sj}$.

The term $F_j$ is the proportion of category j of submarines lost in a battle of duration t and is a consequence of the average loss rate of a category j of submarines being $(B_j + N_{.j} \bar{K}_j)/T_{cj}$. See for this Eqs. B.7, B.11, B.16.

The term $P_j$ is the expected number of patrols made by a submarine of category j in a battle of certain duration t. A battle of very long duration gives an expected number of patrols (Eq. B.18 for $t \to \infty$): the reciprocal of $B_j + N_{.j} \bar{K}_j$ .

It is possible to derive the number of submarine patrols in another way. Therefore we consider that the probability that a submarine is killed in a patrol is $B_j + N_{.j} \bar{K}_j$. In a time t there will be $t/T_c$ patrols. The probability that a submarine survives this number of patrols is:

$$[1 - (B_j + N_{.j} \bar{K}_j)]^{t/T_c} .$$

Assuming $p = B_j + N_{.j} \bar{K}_j$, the expected number of patrols in time $t$ will be:

$$F'_j = \frac{1}{T_c} \int_0^t (1-p)^{v/T_c}\, dv = \frac{1}{\ln(1-p)} [-1+(1-p)^{t/T_c}] . \qquad \text{[Eq. B.19]}$$

For small values of $p$ the terms in Eq. B.19 can be changed by:

$$\ln(1-p) \approx -p$$

and

$$(1-p)^{t/T_c} = \exp[t/T_c \ln(1-p)] \approx \exp(-\frac{pt}{T_c}) ,$$

and so Eq. B.19 can be rewritten as

$$F'_j \approx \frac{1}{p}\left\{1 - e^{-\frac{pt}{T_c}}\right\} = F_j$$

where

$$p = B_j + N_{.j} \bar{K}_j .$$

# APPENDIX C

## ESTIMATION OF ENGAGEMENT RATES AND ENGAGEMENT OUTCOMES

The Lanchester Differential Equations, Eqs. B.2, B.3 and B.4, depend strongly on engagement rates ($\lambda$'s) and engagement outcomes (M's, E's and K's). The purpose of this appendix is to discuss the meaning of these terms and provide methods for estimating them.

### C.1 Definitions

An engagement by a submarine is defined as one (or both) of the following events:

1. A target, which may be a convoy or an independent ship, is detected, closed and attacked by the submarine, i.e. weapons are fired at the target.

2. The submarine, in attempting to attack a target, is sunk.

According to this definition, no engagement has occurred if the submarine, in attempting to attack a target, fails to complete the attack (i.e. is dissuaded from firing weapons) but is not sunk.

Each engagement reduces the weapon supply of each attacking submarine, and the number of engagements can be used for estimating both the probability of a submarine being weapon-limited and the expected time that a submarine is on patrol.

The broader event called an encounter, which is considered in Sect. C.3, includes not only actual engagements but also possible engagements that do not materialize.

When a submarine engages a protected convoy, merchant ships, escorts or submarines may be sunk, and it is convenient to let the expected number of merchant ships, escorts, and submarines sunk per engagement be denoted by M, E and K, respectively. Since an engagement involves only one submarine, K may be interpreted as the probability that a submarine is sunk in an engagement.

It is worth noting that a single engagement with a convoy may lead to attacks against several ships. This is particularly true of engagements between nuclear submarines and slow convoys.

As was mentioned in Appendix A, engagements are generated only by disengaged submarines on patrol. Therefore, the rate of occurrence of engagements, denoted by $\lambda$, is defined on the basis of one disengaged submarine on patrol.

## C.2 Estimation of the Engagement Rate, $\lambda$

The engagement rate per disengaged submarine on patrol, $\lambda$, equals the product of:

a. The rate at which targets enter the submarine's patrol area, r.

b. The probability that the submarine detects, closes and engages a target, e:

$$\lambda = re \qquad \text{[Eq. C.1]}$$

The target rate, r, is zero of course for all classes of targets that are not routed through the submarine's patrol area.

When the target class considered is a convoy that passes through the submarine's patrol area for both the outbound and return journey, then the target rate is given by:

$$r = \frac{2}{\theta} \qquad \text{[Eq. C.2]}$$

where $\theta$ is the convoy sailing interval.

When the target class considered is independent ships the target rate should satisfy the following equation:

$$r = P_s \frac{2X}{\bar{t}} , \qquad \text{[Eq. C.3]}$$

where $X$ is the number of independent merchant ships in the system,

$\bar{t}$ is the average round trip time of these ships,

$P_s$ is the probability that a target routed through the submarine patrol area survives long enough on a given ocean transit to enter this area.

The probability that a submarine detects, closes and engages a target can be estimated by:

$$e = \frac{W}{L} \cdot Cg \qquad \text{[Eq. C.4]}$$

where $W$ = the sweep width for target detection and classification,

$L$ = the width of uncertainty of the shipping lane at the time of detection,

$C$ = the probability of closure given detection,

$g$ = the probability that a submarine fails to make an engagement, given that it can close the target.

In Ref. 6 it is shown that in most situations of interest the probability of closure given detection is approximated by

$$C = \min(\frac{u}{v}, 1) , \qquad \text{[Eq. C.5]}$$

where $u$ is the submarine effective closure speed and $v$ is the target speed.

The probability $g$ is less than 1 because some submarines are dissuaded from attacking targets due to the presence of defensive forces, and others fail to make an engagement due to errors or malfunctions. The value of $g$ can be estimated by means of a detailed encounter flow diagram.

## C.3 Encounter Flow Diagrams, and the Estimation of g, M, K and E

Values of $g$, $M$, $K$ and $E$ can be estimated using encounter flow diagrams, examples of which appear in Refs. 4 and 7. Using these diagrams as a guide, one can readily obtain algebraic expressions for $g$, $M$, $K$ and $E$ in terms of the fundamental transition probabilities and expected values appearing in the diagram. Although this procedure is straightforward, it is also rather lengthy.

## APPENDIX D

## LIST OF SYMBOLS

Remarks on indices:

$d_{ij}$ gives a relation between targets of category $i$ and submarine of category $j$

$d_{.j}$ holds for submarines of class $j$ and is a summation over all target categories

$$d_{.j} = \sum_{i=1}^{I} d_{ij} .$$

$\bar{d}_j$ holds for submarines of class $j$ and is a weighted mean over the target categories

$$\bar{d}_j = \sum_{i=1}^{I} \lambda_{ij} \; d_i / \lambda_{.j} .$$

$d_j$ holds only for submarines of class $j$.

### LIST OF SYMBOLS

$B_j$ The probability that a submarine is destroyed by the indirect defence in a submarine cycle.

E Average number of escorts destroyed per engagement by a submarine.

F Percentage of number of submarines survived a certain time.

h(v) The probability that the submarine is on patrol at time v after starting his patrol.

I Number of categories of targets.

J Number of categories of submarines.

K Probability that a submarine is destroyed in an engagement.

$\ell$ Number of attacks planned for in a patrol.

M Number of merchant ships sunk per engagement.

m Average losses of merchant ships per submarine in a certain time.

| | |
|---|---|
| $N$ | Average number of engagements per submarine patrol. |
| $\tilde{N}$ | Average number of engagements per submarine patrol when the submarine is endurance limited. |
| $P$ | Average number of submarine patrols at a certain time. |
| $S$ | Expected number of disengaged submarines on patrol at a certain time. |
| $S_p$ | Average number of submarines on patrol at a certain time. |
| $T_c$ | Average cycle time of submarines. |
| $T_m$ | Maximum patrol time of a submarine. |
| $\bar{T}_p$ | Average patrol time of a submarine. |
| $T^*_s$ | Disengaged time in a patrol when the submarine is endurance limited. |
| $\bar{T}_s$ | Average disengaged time per patrol. |
| $t$ | Time. |
| $x$ | The expected number of ship losses in a certain time. |
| $y$ | The expected number of submarine losses in a certain time. |
| $Y$ | Number of submarines survived for a certain time. |
| $Y_0$ | Initial number of submarines. |
| $z$ | The expected number of escorts lost in a certain time. |
| $\lambda$ | Average number of engagements per day searching between categories of targets and submarines. |
| $\lambda^1$ | Transformed engagement rate, which gives the average number of complete engagements (viz. including search and engagement time) per day. |
| $\mu$ | Arrival rate of submarines in the patrol area at a certain time. |
| $\tau$ | Engagement time between targets and submarines. |

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